Tutorial 3

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1. **Example 2 on P35: The Plucked String**
   For a vibrating string with the speed c, consider an infinitely long string which satisfies the wave eqution:
   $$\partial_t^2 u - c\partial_x^2 u = 0 \qquad -\infty < x < \infty$$
   Soppose that the the initial position is
   $$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & |x| < a \\ 0 & |x| > a \end{cases}$$
   and the initial velocity $\psi(x) = 0$ for all x.

   The solution of this initial value problem by d'Alembert Formula is
   $$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)]$$
   See the Figure 2 on Page 36.

   The effect of the initial position $\phi(x)$ results a pair of travelling waves, one to the left $\frac{1}{2}\phi(x+ct)$ and another to the right $\frac{1}{2}\phi(x-ct)$, at the speed c and with half the original amplitude $\frac{b}{2}$. You can see this phenomenen by the graphs on page 36 clearly.

2. Let $\phi(x) = 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile at each of the successive instants $t = \frac{a}{2c}, \frac{a}{c}, \frac{3a}{2c}, \frac{2a}{c}, \frac{5a}{c}$.

   Solution: By d'Alembert's formula, the solution is
   $$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}[\text{length of } (x-ct, x+ct) \cap (-a,a)].$$
   So we have
   $$u(x, a/2c) = \begin{cases} 0 & x \in (-\infty, -\frac{3a}{2}] \cup [\frac{3a}{2}, \infty); \\ \frac{1}{2c}(\frac{3a}{2} - x) & x \in [\frac{a}{2}, \frac{3a}{2}]; \\ \frac{a}{2c} & x \in [-\frac{a}{2}, \frac{a}{2}]; \\ \frac{1}{2c}(\frac{3a}{2} + x) & x \in [-\frac{3a}{2}, -\frac{a}{2}]; \end{cases}$$
   $$u(x, a/c) = \begin{cases} 0 & x \in (-\infty, -2a] \cup [2a, \infty \\ \frac{1}{2c}(2a - x) & x \in [0, 2a]; \\ \frac{1}{2c}(2a + x) & x \in [-2a, 0]; \end{cases}$$
   $$u(x, 3a/2c) = \begin{cases} 0 & x \in (-\infty, -\frac{5a}{2}] \cup [\frac{5a}{2}, \infty); \\ \frac{1}{2c}(\frac{5a}{2} - x) & x \in [\frac{a}{2}, \frac{5a}{2}]; \\ \frac{a}{c} & x \in [-\frac{a}{2}, \frac{a}{2}]; \\ \frac{1}{2c}(\frac{5a}{2} + x) & x \in [-\frac{5a}{2}, -\frac{a}{2}]; \end{cases}$$
   $$u(x, 2a/c) = \begin{cases} 0 & x \in (-\infty, -3a] \cup [3a, \infty \\ \frac{1}{2c}(3a - x) & x \in [a, 3a]; \\ \frac{a}{c} & x \in [-a, a]; \\ \frac{1}{2c}(3a + x) & x \in [-3a, -a]; \end{cases}$$

$$
u(x,5a/c) = \begin{cases} 0 & x \in (-\infty,-6a] \cup [6a,\infty); \\ \dfrac{1}{2c}(6a-x) & x \in [4a,6a]; \\ \dfrac{a}{c} & x \in [-4a,4a]; \\ \dfrac{1}{2c}(6a+x) & x \in [-6a,-4a]; \end{cases}
$$

Here we omit the figures. $\square$

3. Stability for diffusion equation by maximum principle

Thoerem: Soppose $u_i(x,t), i=1,2$ are solutions of the following Initial-Boundary-Value-Problem:

$$
\begin{aligned}
&\partial_t u = k\partial_x^2 u \quad 0 \le x \le l, 0 \le t \le T \\
&u(x,t=0) = \phi_i(x) \quad\ 0 \le x \le l \\
&u(x=0,t) = g_i(t),\ u(x=l,t) = h_i(t) \ \ 0 \le t \le T
\end{aligned}
$$

Then

$$
\max_R |u_1(x,t) - u_2(x,t)| \le \max\{\max_{0\le x\le l} |\phi_1(x)-\phi_2(x)|, \max_{0\le t\le T} |g_1(t)-g_2(t)|, \max_{0\le t\le T} |h_1(t)-h_2(t)|\}
$$

Proof: Set $v(x,t) = u_1(x,t) - u_2(x,t)$, the v satisfys

$$
\partial_t v(x,t) = k\partial_x^2 v(x,t)
$$

$$
v(x,t=0) = \phi_1(x) - \phi_2(x)
$$

$$
v(x=0,t) = g_1(t) - g_2(t),\ v(x=l,t) = h_1(t) - h_2(t)
$$

Apply Maximum Principle to v, we have for any $0 < x < l, 0 < t < T$

$$
\begin{aligned}
v(x,t) \le \max_{\partial_p R} v(x,t) &= \max\{\max_{0\le x\le l} \phi_1(x)-\phi_2(x), \max_{0\le t\le T} g_1(t)-g_2(t), \max_{0\le t\le T} h_1(t)-h_2(t)\} \\
&\le \max\{\max_{0\le x\le l} |\phi_1(x)-\phi_2(x)|, \max_{0\le t\le T} |g_1(t)-g_2(t)|, \max_{0\le t\le T} |h_1(t)-h_2(t)|\}
\end{aligned}
$$

Apply Minimum Principle to v, we have for any $0 < x < l, 0 < t < T$

$$
v(x,t) \ge \min_{\partial_p R} v(x,t) = \min\{\min_{0\le x\le l} \phi_1(x)-\phi_2(x), \min_{0\le t\le T} g_1(t)-g_2(t), \min_{0\le t\le T} h_1(t)-h_2(t)\}
$$

Then

$$
\begin{aligned}
-v(x,t) &\le \max\{\max_{0\le x\le l} \phi_2(x)-\phi_1(x), \max_{0\le t\le T} g_2(t)-g_1(t), \max_{0\le t\le T} h_2(t)-h_1(t)\} \\
&\le \max\{\max_{0\le x\le l} |\phi_1(x)-\phi_2(x)|, \max_{0\le t\le T} |g_1(t)-g_2(t)|, \max_{0\le t\le T} |h_1(t)-h_2(t)|\}
\end{aligned}
$$

Hence

$$
\max_R |v(x,t)| \le \max\{\max_{0\le x\le l} |\phi_1(x)-\phi_2(x)|, \max_{0\le t\le T} |g_1(t)-g_2(t)|, \max_{0\le t\le T} |h_1(t)-h_2(t)|\}.
$$

4. Prove the uniqueness of solution to

$$
\begin{aligned}
&\partial_t u = k\partial_x^2 u \quad 0 \le x \le l, 0 \le t \\
&u(x,t=0) = \phi(x) \\
&\partial_x u(x=0,t) = g(t),\ \partial_x u(x=l,t) = h(t)
\end{aligned}
$$

by energy method.

Proof: Suppose u_1 and u_2 are two solutions of the above problem, then $v = u_1 - u_2$ satisfies

$$\begin{aligned}
&\partial_t v = k\partial_x^2 v \quad 0 \le x \le l, 0 \le t \\
&v(x, t = 0) = 0 \\
&\partial_x v(x = 0, t) = 0,\ \partial_x v(x = l, t) = 0
\end{aligned}$$

Multiplying $\partial_t v = k\partial_x^2 v$ by v and then integrating w.r.t x give that

$$\frac{d}{dt}\int_0^l \frac{1}{2}|v|^2 dx = \int_0^l k\partial_x^2 v v dx$$

It follows from integration by parts and boundary conditions that

$$\frac{d}{dt}\int_0^l \frac{1}{2}|v|^2 dx = k\partial_x v v\Big|_0^l - \int_0^l k(\partial_x v)^2 dx = -\int_0^l k(\partial_x v)^2 dx \le 0$$

for any $t \ge 0$. Then for any $t \ge 0$

$$\int_0^l \frac{1}{2}|v|^2 dx \le \int_0^l \frac{1}{2}|v(x, 0)|^2 dx = 0$$

due to the initial condition. Hence $v \equiv 0$ for any $0 \le x \le l, t \ge 0$ which completes the proof of the uniqueness.